Übungen zur Mathematik 1
Lösungen Blatt 13

Aufgabe 1

a) $(4x^5)' = 4\cdot 5\, x^4 = 20x^4$

b) $(2x^{a+1})' = 2(a+1)\,x^a$

c) $\left(\sqrt[4]{x^3}\right)' = \left(x^{\frac34}\right)' = \frac34 x^{\frac34-1} = \frac34 x^{-\frac14} = \frac{3}{4\cdot\sqrt[4]{x}}$

d) $\left(\frac{x^2}{\sqrt[3]{x}}\right)' = \left(x^2\cdot x^{-\frac13}\right)' = \left(x^{\frac53}\right)' = \frac53 x^{\frac23} = \frac53\sqrt[3]{x^2}$

e) $\left(\sqrt[3]{x^4}\right)' = \left(x^{\frac43}\right)' = \frac43 x^{\frac13} = \frac43\sqrt[3]{x}$

f) $\left(x^{\frac12}\right)' = \frac12 x^{-\frac12} = \frac{1}{2\sqrt{x}}$ ← merken!

Aufgabe 2

a) $(-10x^4+2x^3-2)' = -40x^3+6x^2$

b) $(a\cdot\cos x - x^2 + e^x + 1)' = -a\cdot\sin x - 2x + e^x$

c) $\left(\frac{10}{x^3} - 3\ln x + \tan x\right)' = -\frac{30}{x^4} - \frac{3}{x} + \frac{1}{\cos^2 x}$

d) $\left(4\cdot\underbrace{\sqrt[3]{x^5}}_{x^{\frac53}} - 4e^x + \sin x\right)' = 4\cdot\frac53 x^{\frac23} - 4e^x + \cos x$
$= \frac{20}{3}\sqrt[3]{x^2} - 4e^x + \cos x$

Aufgabe 3

a) $\left((4x^3-2x+1)\cdot(x^2-2x+5)\right)'$
$= (12x^2-2)(x^2-2x+5) + (4x^3-2x+1)(2x-2)$
$= 20x^4 - 32x^3 + 54x^2 + 10x - 12$

b) $(\underbrace{\tan x\cdot\tan x}_{\tan^2 x})' = \frac{1}{\cos^2 x}\cdot\tan x + \tan x\cdot\frac{1}{\cos^2 x}$
$= \frac{2}{\cos^2 x}\cdot\frac{\sin x}{\cos x} = \frac{2\sin x}{\cos^3 x}$

c) $(\sin x\cdot\cos x)' = \cos x\cdot\cos x - \sin x\sin x$
$= \cos^2 x - \sin^2 x$

d) $\left((3x+5x^2-1)(3x+5x^2-1)\right)'$
$= (3+10x)(3x+5x^2-1) + (3x+5x^2-1)(3+10x)$
$= 2(3+10x)(3x+5x^2-1)$
$= 100x^3 + 90x^2 - 2x - 6$

e) $(2x\cdot\ln x)' = 2\ln x + 2\cancel{x}\cdot\frac{1}{\cancel{x}} = 2\ln x + 2$

f) $(e^x\cdot\cos x)' = e^x\cos x - e^x\sin x = e^x(\cos x - \sin x)$

g) $(x^n\cdot e^x)' = nx^{n-1}e^x + x^n e^x = x^{n-1}e^x(n+x)$

h) $(\ln x\cdot\cosh x)' = \frac1x\cdot\cosh x + \ln x\cdot\sinh x$

i) $(x^2\cdot\arcsin x)' = 2x\arcsin x + x^2\cdot\frac{1}{\sqrt{1-x^2}}$

j) $(\underbrace{2x}\cdot\underbrace{e^x\cdot\cos x})' = 2e^x\cdot\cos x + 2x(e^x\cdot\cos x)'$
$= 2e^x\cos x + 2xe^x\cos x - 2xe^x\sin x$
$= 2e^x(\cos x + x\cos x - x\sin x)$

Aufgabe 4

a) $\left(\frac{5x^5-6x^2+1}{x^2+2x+1}\right)' = \frac{(25x^4-12x)(x^2+2x+1)-(5x^5-6x^2+1)(2x+2)}{(x^2+2x+1)^2}$

b) $\left(\frac{10x}{x^2+1}\right)' = \frac{10(x^2+1)-10x\cdot 2x}{(x^2+1)^2} = \frac{-10x^2+10}{(x^2+1)^2}$

c) $\left(\frac{\ln x}{x^2}\right)' = \frac{\frac1x\cdot x^2 - \ln x\cdot 2x}{x^4} = \frac{x-2x\ln x}{x^4} = \frac{1-2\ln x}{x^3}$

d) $\left(\frac{2x^3-6x^2+x-3}{x^3-5x}\right)'$
$= \frac{(6x^2-12x+1)(x^3-5x)-(2x^3-6x^2+x-3)(3x^2-5)}{(x^3-5x)^2}$

e) $\left(\frac{\ln x}{e^x}\right)' = \frac{\frac1x\cancel{e^x} - \ln x\cdot\cancel{e^x}}{\cancel{e^x}\cdot e^x} = \frac{\frac1x - \ln x}{e^x} = \frac{1-x\ln x}{xe^x}$

f) $\left(\frac{x^{\frac12}-x^2}{x^2+1}\right)' = \frac{(\frac12x^{-\frac12}-2x)(x^2+1)-(x^{\frac12}-x^2)\cdot 2x}{(x^2+1)^2}$
$= \frac{-\frac32x^{\frac32}-2x+\frac12x^{-\frac12}}{(x^2+1)^2}$

g) $(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2 x-\cos^2 x}{\sin^2 x} \overset{\sin^2 x+\cos^2 x=1}{=} \frac{-1}{\sin^2 x}$

$(\cot x)' = \frac{-1}{\sin^2 x}$ merken!

h) $(\tanh x)' = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \overset{\cosh^2 x-\sinh^2 x=1}{=} \frac{1}{\cosh^2 x}$

i) $\left(\frac{1+\cos x}{1-\sin x}\right)' = \frac{-\sin x(1-\sin x)+(1+\cos x)\cos x}{(1-\sin x)^2}$
$= \frac{\cos x - \sin x + 1}{(1-\sin x)^2}$ wegen $\sin^2 x+\cos^2 x=1$

Aufgabe 5

a) $\left(5(4x^3-x^2+1)^5\right)' = 25(4x^3-x^2+1)^4\cdot(12x^2-2x)$
$= 50(6x^2-x)(4x^3-x^2+1)^4$

b) $\left(\frac{10}{x^3-2x+5}\right)' = \frac{-10}{(x^3-2x+5)^2}\cdot(3x^2-2)$

c) $(\sin(x+2))' = \cos(x+2)\cdot 1$

d) $\left(2\cdot\cos(10x-\frac{\pi}{3})\right)' = -2\sin(10x-\frac{\pi}{3})\cdot 10 = -20\cdot\sin(10x-\frac{\pi}{3})$

e) $(3\cdot e^{-4x})' = 3\cdot e^{-4x}\cdot(-4) = -12\cdot e^{-4x}$

f) $(\sin^2(2x-4))' = 2\sin(2x-4)\cdot\cos(2x-4)\cdot 2$
$= 4\sin(2x-4)\cos(2x-4)$

g) $\left(2\cdot\ln(x^3-2x)\right)' = 2\cdot\frac{1}{x^3-2x}\cdot(3x-2)$

h) $\left(e^{x^2-2x+5}\right)' = e^{x^2-2x+5}\cdot(2x-2)$

i) $\left(\arccos\sqrt{x^2-1}\right)' = \dfrac{-1}{\sqrt{1-\left(\sqrt{x^2-1}\right)^2}}\cdot\dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}$

$$= -\frac{x}{\sqrt{(2-x^2)(x^2-1)}}$$

j) $\left(\arctan(x^2+1)\right)' = \dfrac{2x}{1+(x^2+1)^2}$

k) $\left((x^2-4x+10)^{\frac{2}{3}}\right)' = \dfrac{2}{3}(x^2-4x+10)^{-\frac{1}{3}}\cdot(2x-4)$

$$= \frac{4}{3}\frac{x-2}{\sqrt[3]{x^2-4x+10}}$$

l) $\left((x^3-4x+5)^{-\frac{5}{3}}\right)' = -\dfrac{5}{3}(x^3-4x+5)^{-\frac{8}{3}}(3x^2-4)$

Aufgabe 6

a) $f(x) = x+2x^3$

$f'(x) = 1+6x^2 > 0$ für alle $x\in\mathbb{R}$

$\Rightarrow f$ streng monoton wachsend.

b) Es ist $\left(f^{-1}(x)\right)' = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{1+6(f^{-1}(x))^2}$

und

$$(f^{-1})(0) = 0,\ (f^{-1})(3) = 1.$$

Daher

$$(f^{-1})'(0) = \frac{1}{1+6\cdot 0} = 1,$$

$$(f^{-1})'(3) = \frac{1}{1+6\cdot 1} = \frac{1}{7}.$$

Aufgabe 7

a) $f'(x) = -24x^2+24x+18 = 0$

$\Leftrightarrow\ x^2-x-\frac{3}{4} = 0$, $p=-1$, $q=-\frac{3}{4}$

p,q-Formel: $x_1 = \frac{1}{2}-\sqrt{\frac{1}{4}+\frac{3}{4}} = -\frac{1}{2}$

$x_2 = \frac{1}{2}+\sqrt{\frac{1}{4}+\frac{3}{4}} = \frac{3}{2}$

$f''(x) = -48x+24$

$f''(x_1) = f''(-\frac{1}{2}) = 24+24 = 48 > 0$

$\Rightarrow f$ hat in $x_1 = -\frac{1}{2}$ ein (lokales) Minimum

$f(-\frac{1}{2}) = -5$

$f''(x_2) = f''(\frac{3}{2}) = -72+24 = -48 < 0$

$\Rightarrow f$ hat in $x_2 = \frac{3}{2}$ ein (lokales) Maximum

$f(\frac{3}{2}) = 27$

b) $g(x) = x^4-8x^2+16$

$g'(x) = 4x^3-16x = 0$

$\Leftrightarrow\ 4x(x^2-4) = 0$

$\Leftrightarrow\ 4x(x-2)(x+2) = 0$

$\Leftrightarrow\ x=0 \lor x=2 \lor x=-2$

$g''(x) = 12x^2-16$

$g''(0) = -16 < 0$

$\Rightarrow g$ hat in $x=0$ ein (lokales) Maximum

$g(0) = 16$

$g''(2) = 48-16 = 32 > 0$

$\Rightarrow g$ hat in $x=2$ ein (lokales) Minimum

$g(2) = 0$

$g''(-2) = 48-16 = 32 > 0$

$\Rightarrow g$ hat in $x=-2$ ein (lokales) Minimum

$g(-2) = 0$

c) $h(x) = xe^{-x}$

$h'(x) = e^{-x}-xe^{-x} = 0$

$\Leftrightarrow\ 1-x = 0$

$\Leftrightarrow\ x = 1$

$h''(x) = -e^{-x}-(e^{-x}-xe^{-x})$

$= -2e^{-x}+xe^{-x}$

$= (x-2)e^{-x}$

$h''(1) = -1e^{-1} = -\frac{1}{e} < 0$

$\Rightarrow h$ hat in $x=1$ ein (lokales) Maximum

$h(1) = e^{-1} = \frac{1}{e}$

Aufgabe 8

$f(x) = (x-1)e^{-2x}$

$$f'(x) = e^{-2x} - 2(x-1)e^{-2x}$$
$$= (1-2(x-1))e^{-2x}$$
$$= (3-2x)e^{-2x}$$

$$f''(x) = -2e^{-2x} - 2(3-2x)e^{-2x}$$
$$= (-2-2(3-2x))e^{-2x}$$
$$= (-8+4x)e^{-2x}$$

$$f'''(x) = 4e^{-2x} - 2(-8+4x)e^{-2x}$$
$$= (4-2(-8+4x))e^{-2x}$$
$$= (20-8x)e^{-2x}$$

Lokale Extremwerte:

$f'(x) = 0$

$\Leftrightarrow \quad 3-2x = 0$

$\Leftrightarrow \quad 2x = 3$

$\Leftrightarrow \quad x = \frac{3}{2}$

$$f''\left(\tfrac{3}{2}\right) = \left(-8+4\cdot\tfrac{3}{2}\right)e^{-3} = -\frac{2}{e^3} < 0$$

$\Rightarrow$ f hat in $x = \frac{3}{2}$ ein (lokales) Maximum

$$f\left(\tfrac{3}{2}\right) = \left(\tfrac{3}{2}-1\right)e^{-3} = \frac{1}{2e^3} = 0{,}0249\ldots$$

Wendepunkte:

$f''(x) = 0$

$\Leftrightarrow \quad -8+4x = 0$

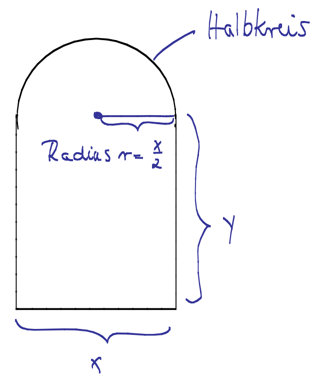
$\Leftrightarrow \quad 4x = 8$

$\Leftrightarrow \quad x = 2$

$$f'''(2) = (20-8\cdot 2)e^{-4}$$
$$= \frac{4}{e^4} \neq 0$$

$\Rightarrow$ f hat in $x=2$ einen Wendepunkt

Aufgabe 9



Umfang $U = x + 2y + \frac{x}{2}\cdot\pi = \left(1+\frac{\pi}{2}\right)x + 2y$

$U = 10$

$\Leftrightarrow \quad \left(1+\frac{\pi}{2}\right)x + 2y = 10$

$\Leftrightarrow \quad y = 5 - \left(\frac{1}{2}+\frac{\pi}{4}\right)x$

Fläche $A = x\cdot y + \frac{1}{2}\left(\frac{x}{2}\right)^2\cdot\pi$

$$= x\left(5-\left(\tfrac{1}{2}+\tfrac{\pi}{4}\right)x\right) + \frac{\pi}{8}\cdot x^2$$
$$= 5x - \left(\tfrac{1}{2}+\tfrac{\pi}{4}\right)x^2 + \frac{\pi}{8}\cdot x^2$$
$$= 5x - \left(\tfrac{1}{2}+\tfrac{\pi}{8}\right)x^2 =: f(x)$$

Fläche wird maximal für x, falls f in x Maximum annimmt.

$f'(x) = 5 - \left(1+\frac{\pi}{4}\right)x = 0$

$\Leftrightarrow \quad \left(1+\frac{\pi}{4}\right)x = 5$

$\Leftrightarrow \quad x = \dfrac{5}{1+\frac{\pi}{4}} = 2{,}8004\ldots$

$$y = 5 - \left(\tfrac{1}{2}+\tfrac{\pi}{4}\right)\frac{5}{1+\frac{\pi}{4}}$$
$$= 5\left(1 - \frac{1+\frac{\pi}{2}}{2+\frac{\pi}{2}}\right)$$
$$= 1{,}4002\ldots$$

Fenster mit der Breite

$x \approx 2{,}8$ m

und der Höhe

$y \approx 1{,}4$ m

hat maximale Fläche bei einem Umfang von 10 m.